On the Query Complexity of Real Functionals

Hugo Férée, Walid Gomaa, Mathieu Hoyrup
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2 Complexity of Norms

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1 Introduction

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Computing with Real Numbers

\[(q_n)_{n \in \mathbb{N}} \sim x \text{ if } \forall n, |x - q_n| \leq 2^{-n}\]  

\[\mathbb{R} \sim (\mathbb{N} \to \mathbb{N})\]  

Model: **Oracle Turing Machines**
- Finite-time computation \(\longrightarrow\) finite number of queries
- \(\longrightarrow\) finite knowledge of the input
- \(\longrightarrow\) continuity

Bounding computation time \(\iff\) bounding
- the computational power
- the number of queries
Computing with Real Numbers

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Model: **Oracle Turing Machines**

Finite-time computation \(\rightarrow\) finite number of queries
\(\rightarrow\) finite knowledge of the input
\(\rightarrow\) continuity

Bounding computation time \(\iff\) bounding

- the computational power
- the number of queries
\[ f \in C[0, 1] \iff \exists \mu, f_Q : \]

modulus of continuity: \( \mu : \mathbb{N} \to \mathbb{N} \)

\[ |x - y| \leq 2^{-\mu(n)} \implies |f(x) - f(y)| \leq 2^{-n} \]

approximation function \( f_Q : \mathbb{Q} \times \mathbb{N} \to \mathbb{Q} \)

\[ |f_Q(q, n) - f(q)| \leq 2^{-n} \]
Computing with Real Numbers Cont’d

Theorem

\[ f : \mathbb{R} \to \mathbb{R} \text{ is computable w.r.t. an oracle } \iff f \text{ is continuous.} \]

Theorem

\[ f : \mathbb{R} \to \mathbb{R} \text{ is polynomial time computable w.r.t. to an oracle } \iff \text{its modulus of continuity is bounded by a polynomial.} \]
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On the Query Complexity of Real Functionals
Dependence of a Norm on a Point

\[ F \text{ is a norm over } C[0, 1] \]

\[ f \in C[0, 1], \quad \alpha \in [0, 1] \]

\[ \Delta \alpha \text{ implies } \Delta F(f) \]
Dependence of a Norm on a Point

Two problems

1. $f$ is continuous, so must change a neighborhood of $\alpha$,

2. if $f = 0$, any change causes $F(f) = 0$ to change to some positive value.
$F$ is always assumed weaker than the uniform norm.

$$d_{F,\alpha}(n) = \sup \{ l : \exists f \in \text{Lip}_1, \text{Supp}(f) \subseteq \mathcal{N}(\alpha, 1/l) \text{ and } F(f) > 2^{-n} \}$$

$d_{F,\alpha} \leq c2^n$, non-decreasing, unbounded.
The uniform norm is monotonic:

$$F = \| . \|_\infty \implies d_{F, \alpha}(n) \sim 2^n$$ (2)
The $L_1$-norm is monotonic:

$$F = \| \cdot \|_1 \quad \Rightarrow \quad d_{F,\alpha}(n) \sim 2^n$$
Some Properties of the Dependency Function

**Proposition**

1. \( \alpha \mapsto d_{F, \alpha}(n) \) is continuous
2. \( F \) is weaker than \( G \implies d_{F, \alpha}(n) \leq d_{G, \alpha}(n + k) \)

Maximal dependence: \( D_F(n) = \max_{\alpha \in [0,1]} d_{F, \alpha}(n) \)

**Proposition**

For \( F \) weaker than the uniform norm:

\[
c_1 2^n \leq D_F(n) \leq c_2 2^n
\]  
(4)
Relevant Points

\[ R_{n,l} = \{ \alpha : d_{F,\alpha}(n) \geq l \} \]

**Definition**

\( \alpha \) is **relevant** if \( \exists c > 0, \forall n, \ d_{F,\alpha}(n) \geq c \cdot 2^n \)

\[ \mathcal{R} = \bigcup_{k} \bigcap_{n} R_{n,2^{n-k}}^{n} \]

**Example**

For \( \| \cdot \|_{\infty} \) and \( \| \cdot \|_{1} \), \( \mathcal{R} = [0, 1] \).
Let $Q = \{q_0, q_1, \ldots, \}$ be some particular canonical enumeration of the dyadic rationals.

Define

$$F(f) = \sum_i 2^{-i} |f(q_i)|$$

Then

$$d_{F,q_i}(n) \geq 2^{n-i}, \quad \text{for } n \geq i$$

$$d_{F,\alpha}(n) \leq \frac{n^2}{\epsilon}$$

$${\mathcal R} = \mathbb{D}$$
Relevant Points

Properties

Theorem
\( \mathcal{R} \) is dense.

Theorem
\( f = 0 \text{ on } \mathcal{R}_{2\mu_f(k)} \implies F(f) \leq c.2^{-k}. \)

Corollary
\( f = g \text{ on } \mathcal{R}_{\mu(k)} \implies |F(f) - F(g)| \leq 2^{-k}. \)
\( \mathcal{R} \) is Dense (Proof)

\[ F(h_0) \geq 2^{-c} \]
$\mathcal{R}$ is Dense (Proof)

\[ F(h_1) \geq 2^{-c-2} \]
$R$ is Dense (Proof)

\[ F(h_2) \geq 2^{-c-4} \]
$R$ is Dense (Proof)

\[
F(h_n) = F(h_{\alpha_n, 2^{-n-p}}) \geq 2^{-c-2n} \implies F(2n + c) \geq 2^{n+p}
\]

$(\alpha_n) \to \alpha$

\[
d_{F, \alpha}(2n + c) \geq 2^{n+p-1} \implies \alpha \in R
\]
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Definition

$Q_n$ : oracle calls of $F$ on $x \mapsto 0$ with precision $2^{-n}$.

Proposition

$R_{n,l} \subseteq \mathcal{N}(Q_{n+1}, \frac{1}{l})$

Definition

$F$ has a polynomial query complexity if it is computable by a relativized OTM with $|Q_n| \leq P(n)$.
Theorem

If $F$ has polynomial query complexity, then almost every point has a polynomial dependency ($d_{F,\alpha} \in \mathcal{P}$ for almost all $\alpha$).

Theorem

If $F$ has polynomial query complexity, then $R$ has Hausdorff dimension 0.

Proposition

$F$ has polynomial query complexity

$\implies \exists \alpha, \frac{2^n}{d_{F,\alpha}(n)}$ is bounded by a polynomial.
Query Complexity
Characterizing polynomial time computable norms

**Theorem**

\[ F \text{ is polynomial time computable w.r.t. an oracle} \iff F \text{ has polynomial query complexity} \iff R_k \text{ can be polynomially covered (wrt. } k). \n\]

\[ R_k = \{ \alpha \in [0, 1]: \forall n, d_{F,\alpha}(n) \geq 2^{\frac{n}{2} - k} \} = \bigcap_n R_{n,\frac{2^n}{2^{n-k}}} \]

**Open question**

Can it be generalized for any \( F : C[0, 1] \rightarrow \mathbb{R} \)?
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Theorem

The following are equivalent:

1. \( F \) is computable by a polynomial time machine doing only one oracle query
2. \( \forall f, F(f) = \phi(f(\alpha)) \) where:
   - \( \alpha \in Poly(\mathbb{R}) \) (but cannot be efficiently retrieved from \( F \))
   - \( \phi \in Poly(\mathbb{R} \rightarrow \mathbb{R}) \)
   - \( \phi \) is uniformly continuous

Open question

Generalization to any finite number of queries?
THANK YOU